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DETERMINATION OF EXPECTED COVERAGE AND OF EXPECTED DAMAGE -- SINGLE BOMB OF LARGE LETHAL AREA

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DETERMINATION OF EXPECTED COVERAGE AND OF EXPECTED DAMAGE — SINGLE BOMB OF LARGE LETHAL AREA

Summary

To apply the photoelectric coverage machine to the determination of the expected coverage when a single bomb of large lethal area is used against an area target, it is only necessary to replace the Gaussian transparency with one whose transparency at each point is proportional to the integral of the Gaussian over the area of an offset circle with center at that point. Under certain conditions this may be approximated by another Gaussian transparency. To determine the expected damage the simple cut-out target mask is replaced with one whose transparency at any point is proportional to the value of the damage achieved if that point falls within the lethal area of the bomb.

Appendices give the moments of the lethal probability function, and its integral over a central circle of given radius.

Foreword

A photoelectric coverage instrument, originally proposed in 1943, has been constructed and prepared for operation (see Reference 1). Although the purpose of this report is to show the application of this instrument to the measurement of expected target coverage and of expected damage value when a target is attacked with a bomb of large lethal area, the mathematical considerations apply to any method of determination.

Vulnerability

The vulnerability of a target is not a physical concept independent of the lethality of the missile. For a given missile, however, one may rationally speak of the relative vulnerability of different parts of the same target.

In an elementary case one may be interested only in knowing what portion of the target area may be destroyed. For example, it may be assumed that all vulnerable portions of the target falling within a given distance of the point of activation of the missile are destroyed, while all other portions are unharmed. In this case the problem reduces to the determination of the expected coverage.

In another case the value of destroying one portion of the target may differ greatly from the value of destroying some other portion of equal area. Where sufficient information is available it may be possible to put a value on the destruction of each element of the target area. If it be assumed that all vulnerable portions of the target which come within the lethal area of the missile are destroyed, then the expected value of destruction is almost as simply determined as is the expected area of destruction.

The treatment of the case where different portions of the target have different vulnerabilities with respect to the same missile will be left to a subsequent paper.

Target Map - Expected Coverage

Where the only question is as to how much of the target area is damaged, and where any portion falling within the lethal area of the missile is assumed to be damaged, the vulnerability of every point in the target area is either 0 or 1. If u and v are map coordinates of the target, the vulnerability is described by the point function $V(u,v)$. The target map as prepared for the photoelectric integrator will be a simple mask, opaque where V is zero, and cut out where V is 1.

Target Map - Expected Damage

If a numerical value can be assigned to the damage which results to each element of the target which falls within the lethal area of the missile, then the value of damage per unit area similarly can be described by a point function $V(u,v)$. In this case V may take on all possible real values, instead of only 0 and 1. (It is recognized that there may be some points of a target whose destruction would be a liability rather than an asset to the attacker, but no attempt will be made here to deal with such a situation in which V takes on negative values.)

Missile Lethality

The lethality of a missile is also a rather meaningless term except when used with respect to a given type of target. For a given target element the relative lethality depends upon the location of that element with respect to the center of activation and the orientation of the missile (in what follows the term "point of impact" may be used in place of the expression "center of activation"). The relative lethality is therefore mathematically a point function with respect to the point of impact. Taking coordinates ξ and η with respect to the point of impact (see Figure 2), the relative lethality may be written $L(\xi, \eta)$.

Dispersion of Impacts

The probability that the actual point of impact (or activation) will fall between x and $x + dx$ and between y and $y + dy$, where x and y are measured relative to the point of aim (or intended point of impact) is similarly a point function of x and y. This probability may be represented by $P(x,y)\,dx\,dy$.

Damage

If, as indicated in Figure 4, impact occurs at the point $u = s$, $v = t$, the damage per unit area at any point (u,v) will be given by the product $V(u,v)\,L(u+s,v+t)$. The quantities $u + s$ and $v + t$ occurring in the second function are the components of displacement of the point (u,v) from the point of impact of the missile, measured in its coordinate system. The total damage value is the product of the damage per unit area times the element of area, integrated over the entire target; i.e.,

$$D = \iint V(u,v)\, L(u+s,v+t)\, du\, dv = D(s,t)\, . \tag{1}$$

This last expression gives the damage if the point of impact is at $u = s$, $v = t$. The probability that the point of impact will lie in a given vicinity depends upon the point of aim. If the point of aim is at $u = a$, $v = b$, then the probability of the point of impact lying in the vicinity of $u = s$, $v = t$, is given by

$$P(s-a,t-b)\, ds\, dt\, . \tag{2}$$

Multiplying this quantity by the damage resulting from this point of impact, and integrating over the target area, the expected damage is given by

$$E(D) = \iint \left\{ P(s-a,t-b) \iint V(u,v)\, L(u+s,v+t)\, du\, dv \right\} ds\, dt\, . \tag{3}$$

Changing the order of integration (which is permissable since the integrals extend over the entire target area) this may be written

$$E(D) = \iint \left\{ V(u,v) \iint P(s-a,t-b)\, L(u+s,v+t)\, ds\, dt \right\} du\, dv\, . \tag{4}$$

The change of variables $s = a + x$ and $t = b + y$ reduces this to

$$E(D) = \iint \left\{ V(u,v) \iint P(x,y)\, L(x+u+a,\, y+v+b)\, dx\, dy \right\} du\, dv\, . \qquad (5)$$

Designating by

$$P_L(s,t) = \iint P(x,y)\, L(x+s, y+t)\, dx\, dy\, , \qquad (6)$$

$$E(D) = \iint V(u,v)\, P_L(u+a, v+b)\, du\, dv \qquad (7)$$

$$= \iint V(u-a, v-b)\, P_L(u,v)\, du\, dv\, . \qquad (7')$$

Application

The photoelectric coverage machine as it is presently constituted will integrate the product of a given set of point functions over an area. Since the immediate interest is in finding the expected damage (or, expected coverage) when a bomb of designated lethal geometry is used subject to a given aiming error (or distribution function) against various targets, the form of the evaluation given by Equation 7 (or 7') appears most useful. Not only are a variety of targets to be investigated, but also the possible complexity of these targets makes it desirable to simplify the construction of the target mask as much as possible.

The point function P_L must be constructed as a separate transparency for each combination of lethality and aiming accuracy.

Bomb with Circular Lethal Area

In a simplified case the assumption is made that the destruction of any vulnerable portion of the target lying within the radius R of the point of activation will be complete, while any portion of the target lying outside that distance will not be damaged. If, in addition, the impact probability of a hit obeys a circular Gaussian distribution law, then the P_L function can be obtained directly from a table of the offset-circle probabilities. This follows because the integral

$$\iint P(x,y)\, L(x+s, y+t)\, dx\, dy$$

$$= \frac{1}{2\pi\sigma^2} \iint e^{-(x^2+y^2)/2\sigma^2}\, L(x+s, y+t)\, dx\, dy \tag{8}$$

where the integration is carried over the entire x,y plane and where L has unit value over the area bounded by a circle of radius R with center at $x = -s$, $y = -t$, and is elsewhere zero. But this is identical with the integral of the circular Gaussian of standard deviation σ carried over a circular area of radius R whose center is at the distance $(s^2 + t^2)^{1/2}$ from the modal point of the distribution. Therefore in this case

$$P_L(s,t) = 1 - q(R/\sigma, \sqrt{s^2 + t^2}/\sigma) \tag{9}$$

where $q(R,x)$ is the probability of missing a circle of radius R when the point of aim is at a distance x from the center of the circle. This function is tabulated in Reference 2; a more extensive tabulation has been prepared jointly by RAND and the Institute for Numerical Analysis (of the Bureau of Standards.

In passing, it may be noted that these P_L transparencies depend on a single parameter, the ratio R/σ. A master plate having been made for each of the desired ratios of R to σ, large or small ratios of aiming error to target dimensions might be obtained photographically.

While the only lethal geometry described here is that of complete destruction within radius R and no destruction outside that radius, it should be noted that any lethal geometry can be handled by the construction of a suitable transparency.

Procedure

The procedure of determining the expected damage may be illustrated with two examples.

Example I. Suppose all the vulnerable portions of the target area have the same damage value per unit area. The process of determining the expected damage is then identical with that used in determining the expected coverage. The target transparency in this case consists of a simple mask, the vulnerable portions of the target being the cut-out portions of the mask.

A calibration reading is made with the desired lethal probability transparency in place. The meter reading, M_c is proportional to the lethal area (A_L) of the bomb; i.e.,

$$M_c = k \iint P_L(x,y)\, dx\, dy = k\, A_L \quad .$$

A second reading is made with both the lethal transparency and the target mask in place, without changing the sensitivity (or the amplification) of the machine. The reading, M_T, now obtained is proportional to the expected coverage: $M_T = k\, E(C)$. The ratio of these meter readings, $M_T/M_C = E(C)/A_L$, is the ratio of the expected coverage to the lethal area of the bomb. This ratio, multiplied by the lethal area and by the damage value per unit area, is the expected damage value.

Lateral motion of the target mask relative to the probability transparency is equivalent to changing the point of aim. The relative **position** giving the greatest meter reading locates the point of aim for the maximum expected damage.

In some cases it may be desirable to construct the lethal probability transparency to so large a scale that the integral of the probability function over the excluded portion is an appreciable portion of the entire integral. In that case the calibration reading made with the probability transparency alone corresponds to an amount less than the lethal area of the bomb. The integral of the probability function over any limited radius is given in Appendix II.

Example II. The value of the damage achieved if the lethal area of the bomb covers one part of the target, will not, in general, be the same as that achieved if it covers another but equal portion of the target area. If the value of the destruction of any element of the target be divided by the area of that element the quotient will be a value-density for that part of the target area. This value-density will be a point function of the target area. Instead of the simple cut-out target map used in the determination of expected coverage, the map now has areas of various degrees of transparency. The transparency of any one area of the target is proportional the the value-density assigned to the destruction of that part of the target. Such a target transparency is indicated in Figure 5.

The calibration reading is made with the lethal probability transparency in place and with a continuous transparency inserted in place of the target map. The meter reading obtained here is proportional to the lethal area of the bomb multiplied by the value-density assigned to that transparency. The calibration transparency is then removed, and the target transparency inserted. The ratio of this meter reading to the calibration reading is equal to the ratio of the expected value of the damage to the product of the lethal area of the bomb by the value-density of the calibration transparency.

Shifting the target transparency with respect to the probability transparency is equivalent to changing the point of sim. This permits determining the expected damage as a point function of the point of aim, and of determining the point of aim which maximizes the expected damage.

It may be objected that the damage to any one part of the target is not necessarily proportional to the portion of that element which falls within the lethal area of the bomb. However, for the present it is proposed to treat any more exact analysis as a refinement to be postponed. It is pointed out that the assumption of a sharply defined lethal radius would seem to be a great crudity, and one which warrants earlier refinement; as pointed out earlier in this paper, the machine can readily accommodate a more involved lethal geometry.

Preparation of the Lethal Probability Transparencies

As shown on pages 6 & 7, the P_L transparencies may be designed with the aid of a table of the probability of hitting an offset circle. The problem of making a plate whose transparency at each point has a pre-assigned value, has not at this date been satisfactorily solved. While it is desirable that a continuous transparency be designed, for the present it may suffice to use a half-tone approximation. By a "half-tone" approximation is here meant a mask which has alternate transparent and opaque sections, with the average ratio of transparent area to total area in any one region proportional to the desired transparency.

An alternative is to approximate the continuous variation of the lethal probability by a step function. In this case the lethal probability transparency will consist of a series of annuli. Each annulus is made of a section of film of measured transparency. The area of each annulus, multiplied by its transparency, is made proportional to the integral of the desired transparency over that area. If the inner and outer radii of any one annulus are indicated by r_1 and r_2 and the transparency of that annulus be indicated by T, then this requires that

$$\pi(r_2{}^2 - r_1{}^2)\,T = k \int_{r_1}^{r_2} \left[1 - q(R/\sigma\,,\ \rho/\sigma)\right]\,2\pi\ \rho\,d\rho$$

$$= k\left[S(R/\sigma\,, r_2/\sigma) - S(R/\sigma\,, r_1/\sigma)\right]$$

where k has any convenient value. The evaluation of $S(R/\sigma, r/\sigma)$ is given in Appendix II.

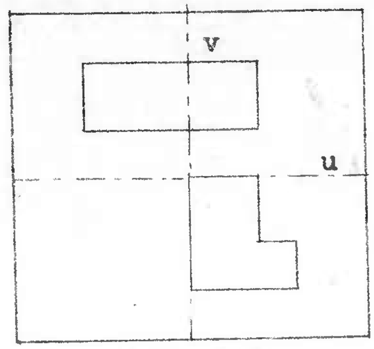

Figure 1

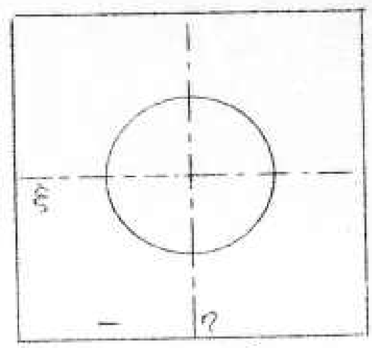

Figure 2

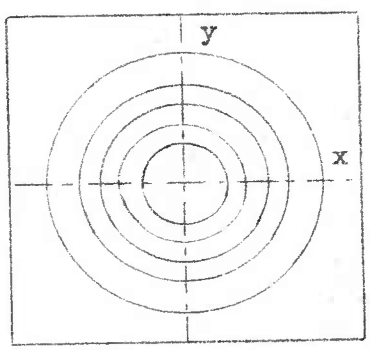

Figure 3

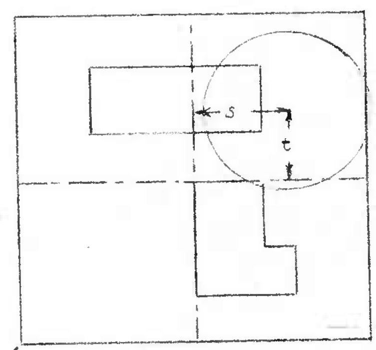

Figure 4

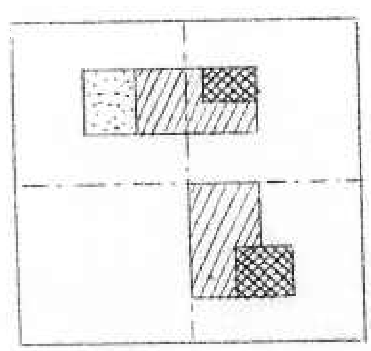

Figure 5

Appendix I

AN APPROXIMATION TO THE LETHAL PROBABILITY TRANSPARENCIES

The integral of the circular Gaussian distribution of standard deviation σ over the area of a circle of radius R with center at a distance x from the mode of the distribution is given by

$$1 - q(R/\sigma, x/\sigma) = \frac{R}{\sigma} e^{-R^2/2\sigma^2} \int_{x/\sigma}^{\infty} e^{-t^2/2} I_1(Rt/\sigma)\, dt \quad .$$

The n^{th} polar moment of this is

$$M_n = \int_0^{\infty} x^n \left[1 - q(R/\sigma, x/\sigma)\right]' 2\pi x\, dx$$

$$= \frac{2\pi R}{n+2} \sigma^{n+1} e^{-R^2/2\sigma^2} \int_0^{\infty} t^{n+2} e^{-t^2/2} I_1(Rt/\sigma)\, dt \quad .$$

The first few even polar moments are:

$$M_0 = \pi R^2$$
$$M_2 = \pi R^2 (2\sigma^2 + R^2/2)$$
$$M_4 = \pi R^2 (8\sigma^2 + 4R^2\sigma^2 + R^4/4)$$
$$M_6 = \pi R^2 (48\sigma^6 + 36R^2\sigma^4 + 6R^4\sigma^2 + R^6/6) \quad .$$

The odd polar moments are

$$M_1 = \frac{\pi R^2}{3\sigma} \sqrt{\frac{\pi}{2}} e^{-R^2/4\sigma^2} \left[(R^2+3\sigma^2) I_0(R^2/4\sigma^2) + (R^2+\sigma^2) I_1(R^2/4\sigma^2)\right]$$

$$M_3 = \frac{\pi R^2}{5\sigma} \sqrt{\frac{\pi}{2}} e^{-R^2/4\sigma^2} \Big[(R^4+9R^2\sigma^2+15\sigma^4) I_0(R^2/4\sigma^2)$$
$$+ (R^4+7R^2\sigma^2+3\sigma^4) I_1(R^2/4\sigma^2)\Big]$$

$$M_5 = \frac{\pi R^2}{7\sigma} \sqrt{\frac{\pi}{2}}\, e^{-R^2/4\sigma^2} \Big[\left(R^6+19R^4\sigma^{-2}+90R^2\sigma^{-4}+105\sigma^{-6}\right) I_0\left(R^2/4\sigma^{-2}\right)$$

$$+ \left(R^6+17R^4\sigma^{-2}+58R^2\sigma^{-4}+15\sigma^{-6}\right) I_1\left(R^2/4\sigma^{-2}\right)\Big] .$$

Further moments can be obtained with the aid of the recurrence relationship

$$M_{n+2} = \frac{n+2}{n+4}\left[R^2 M_n + \sigma^{-3}\left(\frac{2R}{\sigma} - \frac{\sigma}{R}\right)\frac{d}{dR} M_n + \sigma^{-4}\frac{d^2}{dR^2} M_n\right] .$$

The polar moments of a circular Gaussian distribution of variance τ^2 are

$$M_n = \frac{1}{2\pi\tau^2}\int_0^\infty x^n e^{-x^2/2\tau^2}\, 2\pi x\, dx = 2^{n/2}\tau^n\, \Gamma(n/2+1) .$$

In particular,

$$M_0 = 1, \quad M_1 = \tau\sqrt{\pi/2}, \quad M_2 = 2\tau^2, \quad M_3 = 3\tau^3\sqrt{\pi/2},$$

$$M_4 = 8\tau^4, \quad M_5 = 15\tau^5\sqrt{\pi/2}, \quad M_6 = 48\tau^6, \quad \text{etc.}$$

For a Gaussian distribution such that $\tau^2 = \sigma^{-2} + R^2/4$, the even moments are

$$M_2 = 2\sigma^{-2} + R^2/2$$

$$M_4 = 8\sigma^{-4} + 4R^2\sigma^{-2} + R^4/2$$

$$M_6 = 48\sigma^{-6} + 36R^2\sigma^{-4} + 9R^4\sigma^{-2} + 3R^6/4 .$$

These should be compared with the ratios of the corresponding even moments for the offset-circle distribution, namely

$$M_2/M_0 = 2\sigma^2 + R^2/2$$

$$M_4/M_0 = 8\sigma^4 + 4R^2\sigma^2 + R^4/3$$

$$M_6/M_0 = 48\sigma^6 + 36R^2\sigma^4 + 6R^4\sigma^2 + R^6/4 \quad .$$

From this it would appear that when the ratio of the lethal radius, R, to the aiming error, σ, is small, one may approximate the desired lethal probability transparency with a Gaussian transparency whose variance, τ^2, is related to σ^2 and R^2 by $\tau^2 = \sigma^2 + R^2/4$.

The closeness of the approximation for the case where $R = \sigma$ is shown in Figure 6. The moments of the two distributions are compared in Table I.

TABLE I

Comparison of $M_n/M_0\sigma^n$ when $R = \sigma$

Moment	0	1	2	3	4	5	6
Offset Circle	1	1.4038	2.5	5.227	12.33	32.05	90.25
Gaussian	1	1.4012	2.5	5.255	12.50	32.84	93.75

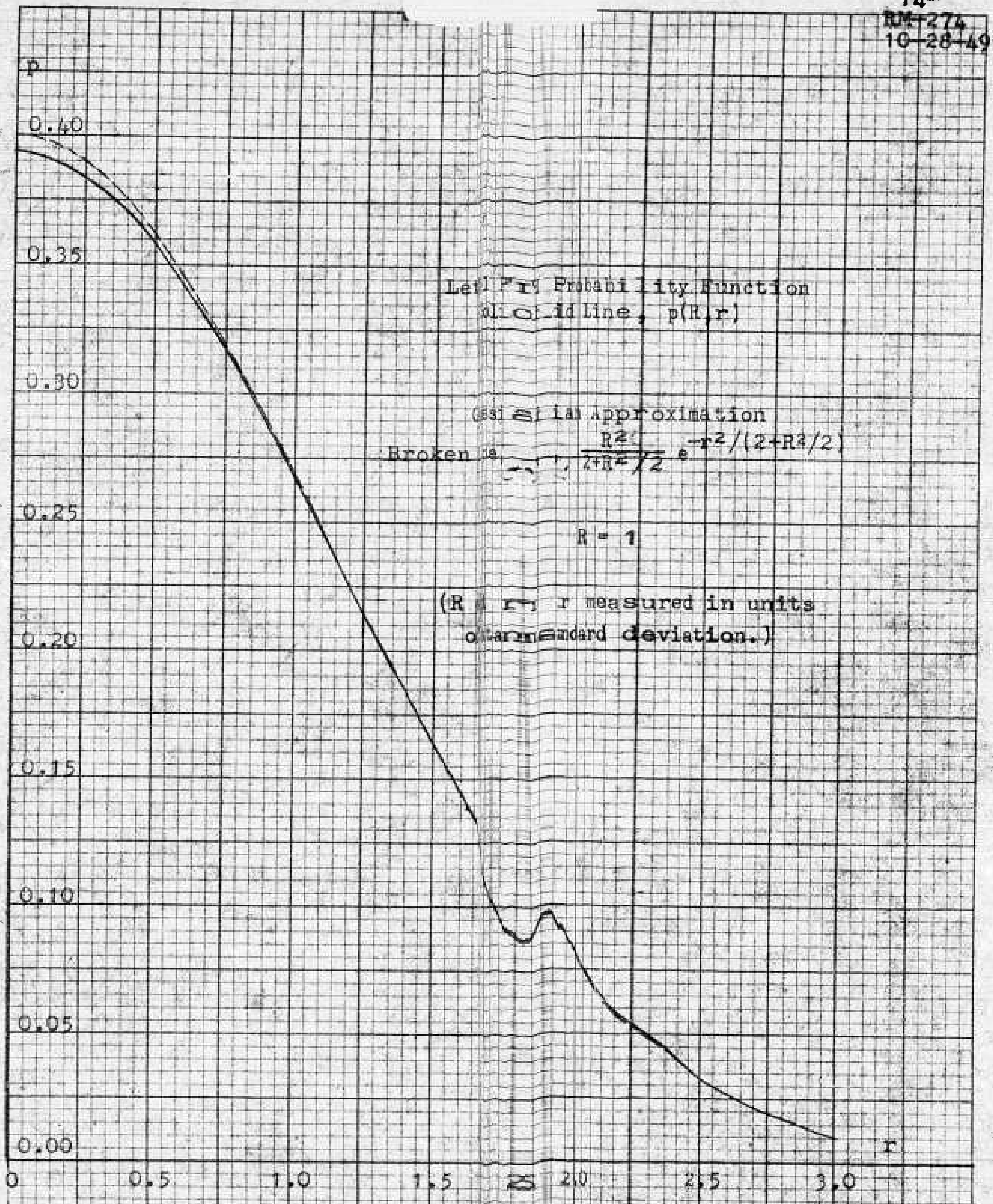

Figure 6. Comparison o[illegible]rrect and Approximate Ordinates of [illegible] Lethal Probability Functions, for the case [illegible] = 1.

Appendix II

INTEGRAL OF THE LETHAL PROBABILITY OVER A CIRCLE OF RADIUS R

The integral of a circular Gaussian distribution of variance σ^2 over the area of a circle of radius R with center at a distance ρ from the mean point of the Gaussian distribution is (see Reference 2)

$$1 - q(R/\sigma, \rho/\sigma) = (R/\sigma)\, e^{-R^2/2} \int_{\rho/\sigma}^{\infty} e^{-t^2/2}\, I_1(Rt/\sigma)\, dt \quad .$$

Consider this as a point function, where ρ is the distance of the point from the origin. The integral of this function over a circle of radius r with center at the origin is

$$S(R/\sigma, r/\sigma) = \int_0^{r/\sigma} \left[1 - q(R/\sigma, \rho/\sigma)\right] 2\pi\, \rho\, d\rho$$

$$= \pi/\sigma^2 \Big\{ R^2 \left[1 - q(r/\sigma, R/\sigma)\right] + r^2 \left[1 - q(r/\sigma, r/\sigma)\right]$$

$$- Rr\, e^{-(R^2+r^2)/2\sigma^2}\, I_1(Rr/\sigma^2) \Big\} \, .$$

Note that

$$S(R/\sigma, r/\sigma) = S(r/\sigma, R/\sigma)$$

$$S(R/\sigma, 0) = 0$$

$$S(R/\sigma, \infty) = \pi R^2/\sigma^2$$

$$S(\infty, r/\sigma) = \pi r^2/\sigma^2$$

$S(R/\sigma, r/\sigma)$ can also be written:

$$S(R/\sigma, r/\sigma) = \pi/\sigma^2 \Big\{ R^2 + (r^2 - R^2)\, q(r/\sigma, R/\sigma)$$

$$- r\, e^{-(R^2+r^2)/2\sigma^2} \Big[r\, I_0(Rr/\sigma^2) + R\, I_1(Rr/\sigma^2) \Big] \Big\}$$

$$= \pi/\sigma^2 \Big\{ r^2 + (R^2 - r^2)\, q(R/\sigma, r/\sigma)$$

$$- R\, e^{-(R^2+r^2)/2\sigma^2} \Big[R\, I_0(Rr/\sigma^2) + r\, I_1(Rr/\sigma^2) \Big] \Big\} .$$

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1. B. T. Himes, "Photoelectric Coverage Machine," RAND Report RM-227, 23 August 1949.

2. H. H. Germond and C. Hastings, Jr., "Scatter Bombing of a Circular Target," Applied Mathematics Panel Report 10.2 R, May 1944.